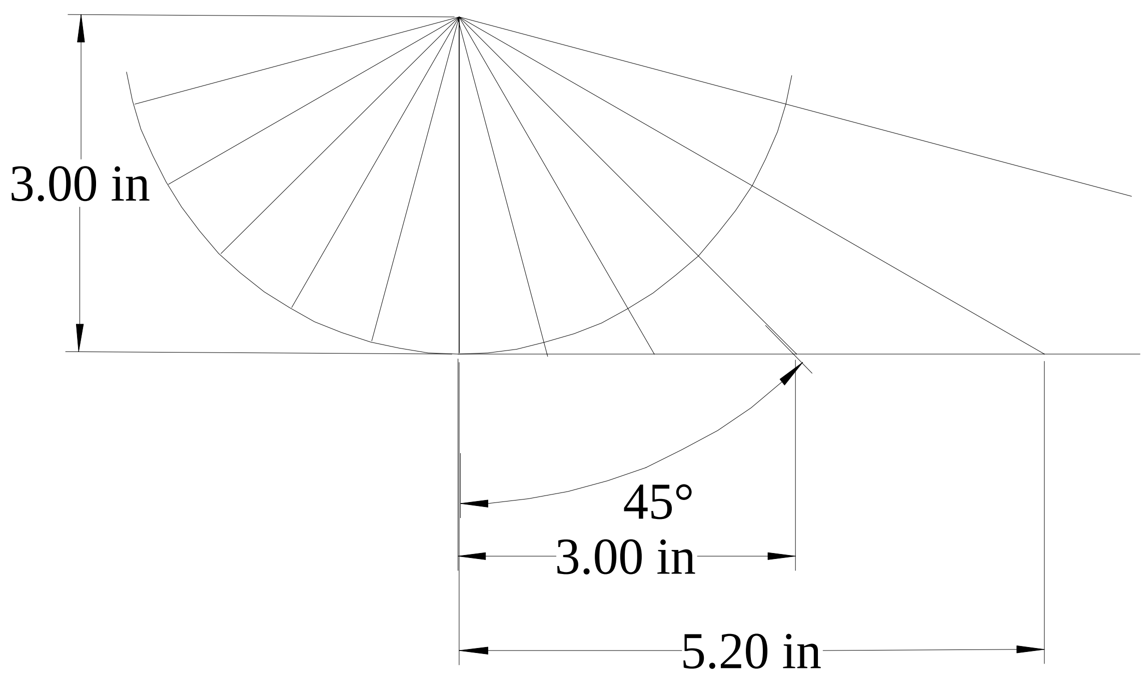
Geometry and Trigonometry, a refresher

If one excluded empirical dial design, and focused on planned dial design, then geometry would be the first line of attack. The following is for flat surfaces, not curved surfaces such as a globe. However, for most purposes we can assume that this works on the surface of the planet.

GEOMETRY

Geometry for simple dials, is a simple process. For example, the polar dial derived from the 15° per hour radials is not involved.

A circle is draw, this represents an equatorial dial, and the 15° radials extended to a horizontal line, this represents the dial plate, in particular, the equinoctial line.



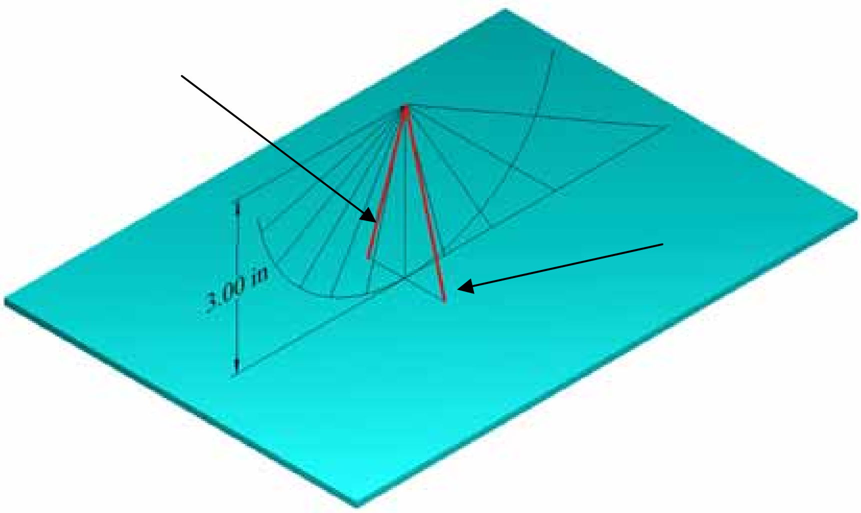
As an aside, the 60° radial shows a distance of 5.2 inches from the bottom of the circle with a radius of 3 inches. The distance along that horizontal line is the trigonometric tangent of the angle, times the radius.

angle	60.00
radians	1.05
tan	1.73
times 3	5.20

Geometry gets slightly more complicated when three dimensions are involved. Something needs to be folded to make the geometric figure two dimensional.

The above polar dial can be used, especially when it comes to lines (equinox) or curves (declination or calendar data).

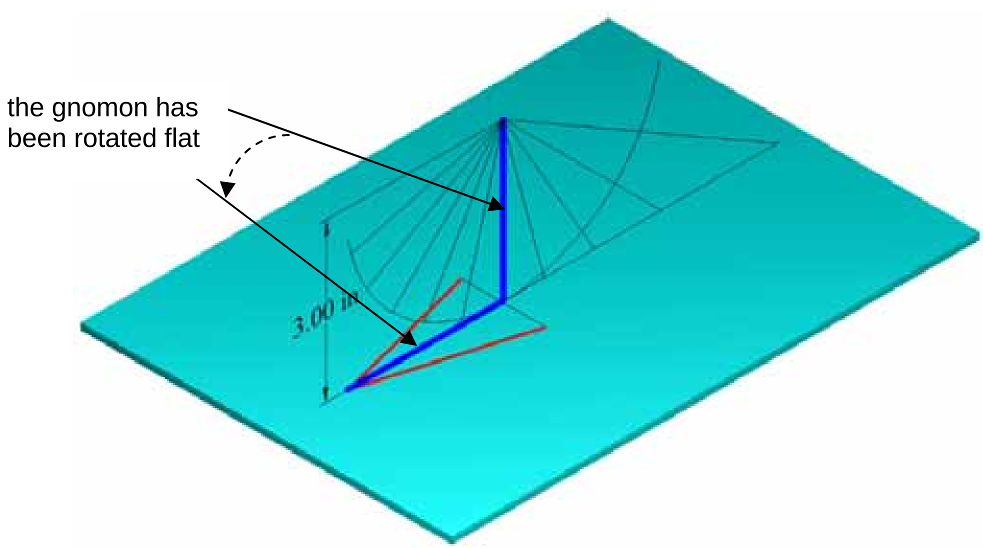
The hour lines are simple enough and were drafted using a flat two dimensional surface. But for the sun's declination to be depicted, a third dimension is added because the rays can hit the gnomon at angles ranging from -23.44 (winter), to 0 (the equinoxes), and +23.44 at the summer solstice.



There are two lines which are 23.44 degrees from the vertical and show the range of the nodus shadow from the winter to summer solstice, with the equinoxes sharing the straight center line.

How can the three dimensional picture above be converted into a two dimensional flat drawing, such that all the relationships are retained.

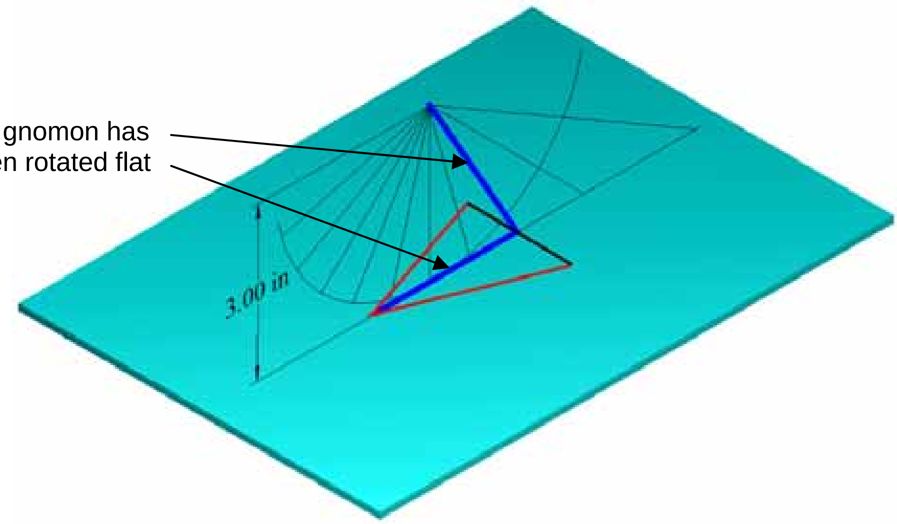
The triangle representing the shadow range can be rotated flat.



After the gnomon has been rotated flat, then the rays of light are projected from the nodus now on the dial plate.

This shows what happens for the solstice range for noon.

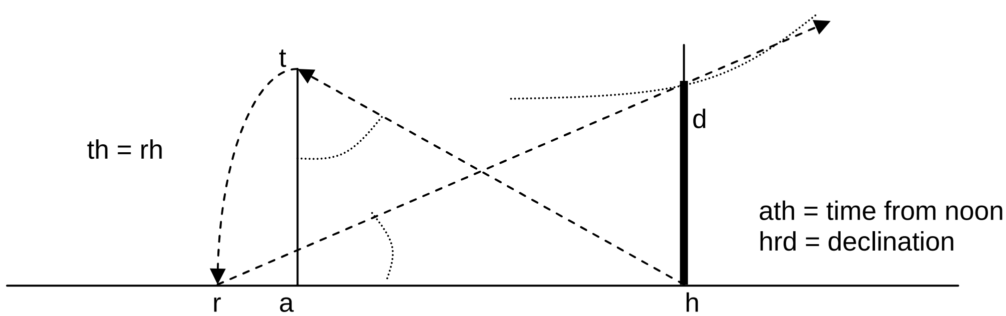
For different hours, that triangle of rays is from the nodus to the hour point on the equinox line.



In this case, two hours from noon (30 degrees from the vertical), results in a nodus to dial plate distance that is longer than the one for the preceding noon example.

The end result is that where the + and – 23.44 degree lines intercept the associated hour line, will form a plot that is hyperbolic.

Convert the above three dimensional pictures into a flat drawing where relationships are retained, the polar dial uses a simple geometric construction, or a simple trigonometric formula.



As an aside, the trigonometric formulae also derive from that picture:-

tan (hrd) = dh / rh = dh / th, thus dh = th * tan (declination) and:-
cos (ath) = ta / th thus th = ta / cos (ath)

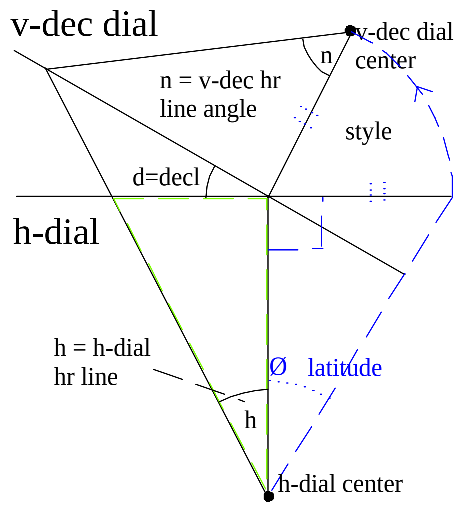
so given: dh = th * tan (declination)
then:- dh = ta * tan (declination) / cos (time)

The distance up the hour line for the point on which the declination (calendar) line will lie is equal to the style linear height times the tangent of the declination all divided by the cosine of the time. This is repeated for several of the hour lines, then the points joined to form a hyperbolic curve for the solstices, or a straight line for the equinox.

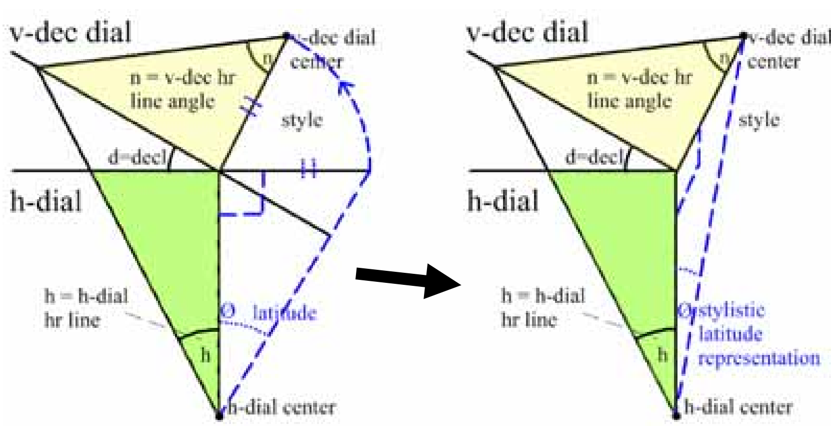
TRIGONOMETRY

Geometry cannot be used as is in programs or spreadsheets. So, the geometric relationships must be converted into things that programs and spreadsheets can use, and that something is trigonometry.

Accurate geometric figures can be used when developing trigonometric formulae, or, they can be stylized.



The same diagram as above can be stylized, as shown below.

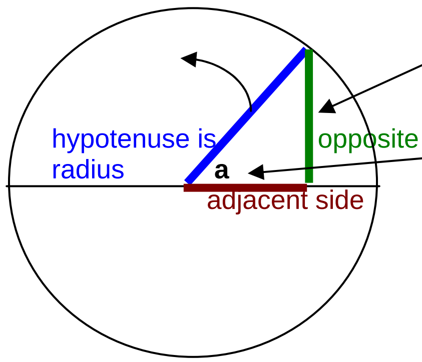


Pure diagram for a vertical decliner retains relationships but has more steps.

Stylized diagram for a vertical decliner has fewer steps, relationships are not retained, but with a little imagination, it is simpler to see how trigonometric formulae are

Trigonometry, as used in these books, uses what are called circular functions.

Circular functions are sin, cos, and tan. They relate the sides of a right angle triangle that is contained in a circle.



the vertical in this triangle constrained in a circle is the "sine" of the angle times the hypotenuse, sine is abbreviated to "sin".

the horizontal in this triangle constrained in a circle is the "cosine" of the angle times the hypotenuse, cosine is abbreviated to "cos".

the cosine is the complement of the sine.

The other way of looking at things is that the:-

sin(a) = opposite / hypotenuse

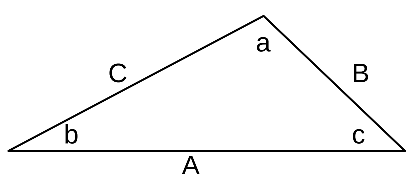
cos(a) = adjacent / hypotenuse

The tangent of that angle, abbreviated "tan": tan(a) = opposite / adjacent

The above functions enable lengths of sides of a triangle to be calculated if the angle "a" is known and the length of one of the sides. The sentence "the old aunt, sat on her, coat and hat" helps remind people of those relationships.

Certain standard relationships exist:- tan = sin / cos

And for any triangle, whether or not it is a right angled triangle,



the law of sines says:-

A / sin(a) = B / sin(b) = C / sin(c)

While normal humans use the "degree" which is 1/360 of a circle, the "radian" is more appropriate for mathematicians, there are 2 * 3.1416 radians in a circle.

The sin, cos, and tan can be calculated from a series

```
sin   =    x - (x**3)/3! + (x**5)/5! - (x**7)/7! + . . .
cos   =    1 - (x**2)/2! + (x**4)/4! - (x**6)/6! + . . .
tan   =    x + (x**3)/3  + 2*(x**5)/15 + . . .
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The opposite of sin, cos, and tan are called arcsin, arccos, and arctan, often abbreviated to asin, acos, and atan, or $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$.

The arcsin and arccos can be developed simply from the following:-

```
asin  =     atan(x/sqrt(1-x*x))
acos  =     (3.1416/2) – asin (x)
```

however, atan is more involved. There are two series often found, one only works for a calue of up to 1, which is 45°, which is not very helpful. The other is:-

```
atan  =     (x/(1+x*x)) * ( 1 + (2/3)*(x*x/(1+x*x)) +
            ((2*4)/(3*5))*(x*x/(1+x*x))*(x*x/(1+x*x)) ...
```

is an ATAN series good for x of any value, however with good precision and many terms it may still have increasing errors beyond 84°.

Thus, if a sundial can be reduced to geometry using the projections discussed earlier, then they can be converted to trigonometry. Most such conversions are straight forward. Some are complex.

Different people use different parts of a geometric figure to develop a trigonometric formula.

For example, the standard formula for the hour line angles of an inclined decliner differs from the formula derived in this series of books.

The results are the same for all ranges of numbers. In fact, one formula can be converted to the other.

This entire series of books, booklets, and supplements, was developed with no spherical trigonometry. In spherical trigonometry, biangles can have angles other than 0°, and anlges in a triangle do not add up to 180 either.

Other functions exist that compare to the circular functions of sin, cos, and tan. There are hyperbolic functions called sinh, cosh, and tanh. They use a hyperbolic curve rather than a circular arc. They are useful in electronics, however, they are not used in this series of books and booklets.